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On instant blow-up for quasilinear parabolic equations with growing initial data

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We are interested in the existence of the solutions of the parabolic equations with initial data which are not bounded at space infinity.

In [4] Giga and the author considered a nonnegative blowing up solution of the semilinear parabolic equation of the form

$$u_t = \Delta u + f(u), \quad x \in \mathbf{R}^N, t > 0$$

with nonlinear terms f and nonnegative initial data u_0 satisfying that f is positive, nondecreasing and convex in $(0, \infty)$, $\int_1^\infty ds/f(s) < \infty$ and there are sequences $\{x_n\} \subset \mathbf{R}^N$ and $\{r_n\} \subset \mathbf{R}_+$ with $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} r_n \geq 0$ such that

$$\lim_{n\to\infty} \frac{b_n}{r_n^2 f(b_n)} \text{ is small enough}$$

with $b_n = \inf\{u_0(x) : |x - x_n| \leq r_n\}$. They showed that the solutions do not exist even locally in time.

We consider the initial value problem for a quasilinear parabolic equation of the form

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbf{R}^N, t \in (0, T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N. \end{cases} \tag{1}$$

Here we assume that $N \geq 1$, $1 \leq m < p$.

We are interested in the problem whether there is a local-in-time solution of (1) when an initial datum u_0 is continuous and grows at the space infinity, for example $\lim_{|x|\to\infty} u_0(x) = \infty$.

We consider the weak solution u in $\mathbf{R}^N \times [0,T)$ of (1) such that $u \in C(\mathbf{R}^n \times [0,\tau))$ for each $\tau \in (0,T)$, and for any bounded domain $\Omega \in \mathbf{R}^N$ with smooth boundary $\partial\Omega$, $0 < \tau < T$ and nonnegative $\phi(x,t) \in C^{2,1}(\Omega \times [0,T))$ which vanishes on the boundary $\partial\Omega$,

$$\begin{aligned}\int_\Omega u(x,\tau)\phi(x,\tau)dx &- \int_\Omega u(x,0)\phi(x,0)dx \\ &= \int_0^\tau \int_\Omega \{u\partial_t\phi + u^m\Delta\phi + u^p\phi\}dxdt - \int_0^\tau \int_{\partial\Omega} u^m \partial_\nu \phi dSdt, \qquad (2)\end{aligned}$$

where ν denote the outer unit normal to the boundary. Note that the solution of (1) may be nonunique. Define $T^* = T^*(u_0)$ as the supremum of all existence times of these solutions.

In this paper we shall prove that $T^* = 0$ when the initial data u_0 is growing at the space infinity. In other words there is even no local-in-time solution such that for any $\tau > 0$ the weak solution does not exist for $t \in (0,\tau)$. We say this phenomenon $T^* = 0$ an *instant blow-up*. We are able to prove that the instant blow-up occurs for more general initial data u_0.

Theorem. *Assume that $u_0 \in C(\mathbf{R}^N)$ is nonnegative. Assume that there are sequences $\{x_n\}_{n=1}^\infty \subset \mathbf{R}^N$ and $\{r_n\}_{n=1}^\infty \subset \mathbf{R}_+$ with $\lim_{n\to\infty}|x_n| = \infty$ and $\lim_{n\to\infty} r_n \geq 0$ such that*

$$\lim_{n\to\infty} r_n^2 b_n^{p-m} > \frac{1}{\varepsilon} \qquad (3)$$

for some $\varepsilon \in (0, 1/c)$, where $b_n = \inf\{u_0(x) : |x - x_n| \leq r_n\}$ and $c > 0$ is the first eigenvalue of $-\Delta$ in a unit ball with the Dirichlet boundary condition. Then $T^* = 0$, i.e., the instant blow-up occurs provided that only nonnegative solutions are considered.

The proof of Theorem depends on a classical Kaplan's argument [6] to show the existence of blow-up which uses principal eigenfunctions of the Laplace operator with the Dirichlet condition.

In [1] among other results there is one about a sufficient condition on initial data for nonexistence of a local-in-time nonnegative solution for $u_t = \Delta u^m + u^p/(1+|x|)^\alpha$ with $m \geq 1$, $p > 1$ and $\alpha \in \mathbf{R}$. In the case of $\alpha = 0$ the condition leads

$$\sup_{x\in\mathbf{R}^n} \int_{B(x,1)} u_0(y)dy = \infty. \qquad (4)$$

In [1] this is explicitly mentioned for $1 < p < m + 2/N$. However, their proof is still valid for all $p > 1$. By the way their main interest is the existence of solution; for example they proved the local existence when

$$\sup_{x\in\mathbf{R}^N}\int_{B(x,1)} u_0(y)dy < \infty$$

for $1 < p < m + 2/N$. The condition (3) is not included in the condition of their result for $p > m + 2/N$. In fact, if $u_0 \geq b_n$ on $B(x_n, r_n)$, then $\lim_{n\to\infty} b_n r_n^N = \infty$ is a sufficient condition for (4) (not a necessary condition). Our condition leads $\lim_{n\to\infty} r_n^2 b_n^{p-m}$ is large enough. This shows that our condition for $p > m + 2/N$ is not included in their condition.

In [1] they also prove the local existence for $p \geq m + 2/N$ when u_0 fulfills

$$\sup_{x\in\mathbf{R}^n}\int_{B(x,1)} u_0^q(y)dy < \infty$$

for some $q > N(p-m)/2$. In our nonexistence result u_0 satisfies

$$\sup_{x\in\mathbf{R}^n}\int_{B(x,1)} u_0^q(y)dy \geq \lim_{n\to\infty}\int_{B(x_n,1)} u_0^q(y)dy \geq \lim_{n\to\infty} \varepsilon^{-\frac{q}{p-m}} r_n^{N-\frac{2q}{p-m}} = \infty$$

for any $q > N(p-m)/2$, where ε is used in (3).

In [4] Theorem was proved in the case $m = 1$. They studied the instant blow-up by using not only the eigenfunction method in [6] same as this paper but also the energy method in [7] and [2].

In the rest of the paper Theorem will be proved by using the Kaplan's argument [6].

Lemma. *(c.f. [3, Lemma 4.2]) Let v be the solution of the integral equation of the form*

$$v(t) - v(0) = \int_0^t h(v(s))ds \tag{5}$$

in $[0, T_0)$ with h satisfying $h \in C^1[0,\infty)$ and $h' \geq 0$. Let $\tilde{v}$ be a nonnegative measurable function on $[0, T_0)$. Assume that $\tilde{v}$ satisfies

$$\tilde{v}(t) - \tilde{v}(t_0) \geq (\leq) \int_{t_0}^t h(v(s))(s)ds \quad \textit{for} \quad t_0, t \in [0, T_0) \quad \textit{with} \quad t_0 \leq t. \tag{6}$$

Assume that $\tilde{v}(0) \geq (\leq) v(0)$. Then

$$\tilde{v}(t) \geq (\leq) v(t) \quad \textit{for} \quad t \in [0, T_0).$$

Proof. We shall only prove the case $\tilde{v}(t) - \tilde{v}(t_0) \geq \int_{t_0}^{t} \tilde{v}^p(s)ds$ since the proof of the other case is parallel. Since $\tilde{v}(0) \geq v(0)$, the estimate (6) together with (5) yields

$$\tilde{v}(t) - v(t) \geq \int_0^t (h(\tilde{v}(s)) - h(v(s)))ds.$$

By the mean value theorem we observe that

$$\tilde{v}(t) - v(t) \geq \int_0^t c(s)\,(\tilde{v}(s) - v(s))\,ds,$$

where

$$c(s) = \int_0^1 h'\,(\theta v(s) + (1-\theta)\tilde{v}(s))\,d\theta.$$

We set $\psi_\epsilon(t) = \tilde{v}(t) - v(t) + \epsilon$ with $\epsilon > 0$, and observe that $\psi_\epsilon(t)$ satisfies

$$\psi_\epsilon \geq \int_0^t c(s)\psi_\epsilon(s)ds + \epsilon\left(1 - \int_0^t c(s)ds\right).$$

We set

$$t_1 = \sup\left\{t > 0; \int_0^t c(s)ds < \frac{1}{2}\right\}.$$

Then, for $t \in [0, t_1]$ we have

$$\psi_\epsilon(t) \geq \int_0^t c(s)\psi_\epsilon(s)ds + \frac{\epsilon}{2}. \tag{7}$$

We shall argue by contradiction to prove $\psi_\epsilon(t) \geq 0$. Suppose that $\psi_\epsilon(t) < 0$ for some $t \in [0, t_1]$. Then $\psi_\epsilon(\tau) = 0$ for

$$\tau = \inf\{t \in [0, t_1]; \psi_\epsilon < 0\}. \tag{8}$$

This τ must be positive. Indeed, since $\tilde{v}$ is nondecreasing by (6) and v is continuous, $\psi_\epsilon(0) > \epsilon$ implies $\tau > 0$.

Since $\int_0^\tau c(s)\psi_\epsilon(s)ds \geq 0$ and (8) imply $\psi_\epsilon(\tau) \leq 0$, we get a contradiction by (7). We thus proved that

$$\psi_\epsilon(t) \geq 0.$$

Since this holds for all $\epsilon > 0$, we get $\tilde{v}(t) \geq v(t)$ for $t \in [0, t_1]$. (If $\tilde{v}(t) < v(t)$ for some t, there exist $\epsilon > 0$ such that $\psi_\epsilon < 0$ for such t.)

Next, since $\tilde{v}(t) \geq v(t)$ for $t \in [0, t_1]$, we observe that

$$\psi_\epsilon \geq \int_{t_1}^{t} c(s)\psi_\epsilon(s)ds + \epsilon\left(1 - \int_{t_1}^{t} c(s)ds\right).$$

We set

$$t_2 = \sup\left\{t > t_0; \int_{t_1}^{t} c(s)ds < \frac{1}{2}\right\}$$

and observe that

$$\psi_\epsilon \geq \int_{t_1}^{t} c(s)\psi_\epsilon(s)ds + \frac{\epsilon}{2}$$

for $t \in [t_1, t_2]$. By the same argument one can prove $\psi_\epsilon \geq 0$ for all $\epsilon > 0$, and $\tilde{v}(t) \geq v(t)$ for $t \in [t_1, t_2]$.

We repeat this argument and conclude that

$$\tilde{v}(t) \geq v(t)$$

for all $t \in [0, T_0)$. By the same argument, we find if

$$\tilde{v}(t) - \tilde{v}(t_0) \leq \int_{t_0}^{t} \tilde{v}^p(s)ds \quad \text{for} \quad t_0, t \in [0, T_0) \quad \text{with} \quad t_0 \leq t,$$

then

$$\tilde{v}(t) \leq v(t) \quad \text{for} \quad t \in [0, T_0).$$

□

Proof of Theorem. Let $\{r_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be as in Theorem satisfying (3). Set $\lambda_n > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $B(0, r_n)$, and let $\phi_n(x) \geq 0$ denote the corresponding positive eigenfunction normalized by $\int_{B(0,r_n)} \phi_n(x)dx = 1$. By scaling it is easy to observe that

$$\lambda_n = \frac{c}{r_n^2} \tag{9}$$

with c defined in Theorem. Define

$$G_n(t) = \int_{B(x_n, r_n)} u(x,t)\phi_n(x - x_n)dx.$$

Let $\nu_n(x)$ denote the outward unit normal to $B(0,r_n)$ at $x \in \partial B(0,r_n)$. By (2) and the fact that $\phi_n = 0$ and $\partial\phi_n/\partial\nu_n \leq 0$ on $\partial B(0,r_n)$ with the unit normal vector ν_n, we obtain

$$G_n(t) \geq G_n(0) + \int_0^t \int_{B(x_n,r_n)} (-\lambda_n u^m(x,s)\phi(x) + u^p(x,s)\phi(x))dxds.$$

Put

$$h_n(s) = \begin{cases} -\lambda_n s^m + s^p, & s \geq \left(\frac{m\lambda_n}{p}\right)^{\frac{1}{p-m}}, \\ -\lambda_n \left(\frac{m\lambda_n}{p}\right)^{\frac{m}{p-m}} + \left(\frac{m\lambda_n}{p}\right)^{\frac{p}{p-m}}, & 0 \leq s \leq \left(\frac{m\lambda_n}{p}\right)^{\frac{1}{p-m}}, \end{cases} \tag{10}$$

similarly as in [5]. Since h_n is convex, we obtain

$$G_n(t) \geq G_n(0) + \int_0^t h_n(G_n(s))ds. \tag{11}$$

by Jensen's inequality. Let us consider the system of ordinary differential equations

$$\begin{cases} g_n'(t) = h_n(g_n(t)), \\ g_n(0) = G_n(0) \geq b_n. \end{cases} \tag{12}$$

Define $T_{g_n} = \sup\{t \geq 0 : g_n(t) < \infty\}$ and $T_{G_n} = \sup\{t \geq 0 : G_n(t) < \infty\}$. Since g_n satisfies

$$g_n(t) = g_n(0) + \int_0^t h_n(g_n(s))ds,$$

and from Lemma, we obtain $G_n \geq g_n$ and $T_{g_n} \geq T_{G_n}$.

Consider the solutions of (1) with the initial data b_n. The maximal existence times of the solutions denoted by $T^*(b_n)$ is estimated as

$$T^*(b_n) = \int_{b_n}^{\infty} \frac{d\xi}{\xi^p}.$$

Note that $\lim_{n\to\infty} T^*(b_n) = 0$. From (3) we may assume that there exist $n_0 \geq 0$ such that

$$\frac{1}{r_n^2 b_n^{p-m}} < \varepsilon$$

for $n \geq n_0$ and $\varepsilon \in (0, 1/c)$. From (9) we see that

$$\lambda_n b_n^m < c\varepsilon b_n^p,$$

and

$$\lambda_n \xi^m < c\varepsilon \xi^p \tag{13}$$

for $\xi \geq b_n$ and $n \geq n_0$. Since $b_n \geq (m\lambda_n/p)^{1/(p-m)}$ by (13), we have

$$T_{g_n} = \int_{b_n}^{\infty} \frac{d\xi}{h_n(\xi)} = \int_{b_n}^{\infty} \frac{d\xi}{-\lambda_n \xi^m + \xi^p}$$

for $n \geq n_0$ by (13). Thus we see that

$$\frac{T^*(b_n)}{T_{g_n}} = \frac{\int_{b_n}^{\infty} d\xi/\xi^p}{\int_{b_n}^{\infty} d\xi/(-\lambda_n \xi^m + \xi^p)} > \frac{\int_{b_n}^{\infty} d\xi/\xi^p}{\int_{b_n}^{\infty} d\xi/\{(1 - c\varepsilon)\xi^p\}} > 1 - c\varepsilon \tag{14}$$

for $n \geq n_0$. Thus we obtain

$$\lim_{m \to \infty} \frac{T^*(b_n)}{T_{g_n}} \geq 1 - c\varepsilon > 0.$$

Noting that $\lim_{n\to\infty} T^*(b_n) = 0$, we see that $\lim_{n\to\infty} T_{g_n} = 0$. Again we get $T_{G_n} \to 0$ as $n \to \infty$. By the definition of the weak solution we have $T^* = 0$. □

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References

[1] D. Andreucci and E. Di Benedetto, *On the Cauchy problem and initial trances for a class of evolution equations with strongly nonlinear sources*, Ann. Scuola Norm. Sup. Pisa, **18** (1991), 363-441.

[2] J. M. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford Ser. (2) **28** (1977), no. 112, 473–486.

[3] Y. Giga and N. Umeda, *On blow-up at space infinity for semilinear heat equations*, J. Math. Anal. Appl. **316** (2006), no. 2, 538–555.

[4] Y. Giga and N. Umeda, *On instant blow-up for semilinear heat equation with growing initial data*, Methods Appl. Anal. **15** (2008), no. 2, 185–196.

[5] T. Imai and K. Mochizuki, *On blow-up of solutions for quasilinear degenerate parabolic equation*, Publ. Res. Inst. Math. Soc. **27** (1991), 695–709.

[6] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330.

[7] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + \mathcal{F}(u)$*, Arch. Rational Mech. Anal. **51** (1973), 371–386.